The Product Rule: The product of two differentiable functions f and g is itself differentiable. Moreover, the derivative of fg is the first function times the derivative of the second, plus the second function times the derivative of the first.

$$\frac{d}{dx}[f(x)\,g(x)] = f(x)\,g'(x) + g(x)\,f'(x)$$

Proof: $\lim_{\Delta x \to 0} \frac{f(x + \Delta x)\,g(x + \Delta x) - f(x)\,g(x)}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{f(x + \Delta x)\,g(x + \Delta x) - f(x + \Delta x)\,g(x) + f(x + \Delta x)\,g(x) - f(x)\,g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left[f(x + \Delta x)\,\frac{g(x + \Delta x) - g(x)}{\Delta x} + g(x)\,\frac{f(x + \Delta x) - f(x)}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} \left[f(x + \Delta x)\,\frac{g(x + \Delta x) - g(x)}{\Delta x} \right] + \lim_{\Delta x \to 0} \left[g(x)\,\frac{f(x + \Delta x) - f(x)}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} f(x+\Delta x) \bullet \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} + \lim_{\Delta x \to 0} g(x) \bullet \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \boxed{f(x)\,g\,'(x) + g(x)\,f\,'(x)}$$ Q.E.D.

- The Product Rule can be extended over products involving more than two factors.
- $\frac{d}{dx}[f(x)\,g(x)\,f(x)] = f'(x)\,g(x)\,h(x) + f(x)\,g'(x)\,f(x) + f(x)\,g(x)\,h'(x)$.
- $\frac{d}{dx}\left[x^2 \sin x \cos x\right] = 2x \sin x \cos x + x^2 \cos x \cos x + x^2 \sin x\,(-\sin x)$
  $= 2x \sin x \cos x + x^2(\cos^2 x - \sin^2 x)$
- $\frac{d}{dx}\left[(3x - 2x^2)(5 + 4x)\right] = (3x - 2x^2)(4) + (3 - 4x)(5 + 4x) = -8x^2 + 12x - 16x^2 - 8x + 15$
  $= -24x^2 + 4x + 15$ There is also the option of multiplying out the original function, then taking the derivative.
- $\frac{d}{dx}[2x \cos x - 2 \sin x] = 2(x\,(-\sin x) + (1)\cos x) - 2(\cos x) = -2x \sin x + 2 \cos x - 2 \cos x$
  $= -2x \sin x$

The Quotient Rule: The quotient f/g of two differentiable functions f and g is itself differentiable at all values of x for which $g(x) \neq 0$. Moreover, the derivative of f/g is given by the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\,f'(x) - f(x)\,g'(x)}{[g(x)^2]}.$$

Proof: $$\lim_{\Delta x \to 0} \frac{\dfrac{f(x+\Delta x)}{g(x+\Delta x)} - \dfrac{f(x)}{g(x)}}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x) - f(x)g(x+\Delta x)}{\Delta x\, g(x+\Delta x)g(x)}$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+\Delta x)}{\Delta x\, g(x+\Delta x)g(x)}$$

$$= \frac{\displaystyle\lim_{\Delta x \to 0} \frac{g(x)[f(x+\Delta x) - f(x)]}{\Delta x} - \lim_{\Delta x \to 0} \frac{f(x)[g(x+\Delta x) - g(x)]}{\Delta x}}{\displaystyle\lim_{\Delta x \to 0} \left[g(x+\Delta x)g(x)\right]}$$

$$= \frac{g(x)\left[\displaystyle\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}\right] - f(x)\left[\displaystyle\lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}\right]}{\displaystyle\lim_{\Delta x \to 0} \left[g(x+\Delta x)g(x)\right]}$$

$$= \boxed{\frac{g(x)\,f'(x) - f(x)\,g'(x)}{[g(x)]^2}}$$ Q.E.D.

- $y = \dfrac{5x - 2}{x^2 + 1}$. $y' = \dfrac{d}{dx}\left[\dfrac{5x - 2}{x^2 + 1}\right] = \dfrac{(x^2 + 1)(5) - (5x - 2)(2x)}{(x^2 + 1)^2} = \dfrac{5x^2 + 5 - 10x^2 + 4x}{(x^2 + 1)^2} = \dfrac{-5x^2 + 4x + 5}{(x^2 + 1)^2}$

- $y = \dfrac{3 - (1/x)}{x + 5} = \dfrac{3x - 1}{x^2 + 5x}$. $y' = \dfrac{(x^2 + 5x)(3) - (3x - 1)(2x + 5)}{(x^2 + 5x)^2} = \dfrac{-3x^2 + 2x + 5}{(x^2 + 5x)^2}$